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Fermion Schwinger's function for the $SU(2)$-Thirring model

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Abstract

We study the Euclidean two-point function of Fermi fields in the $SU(2)$-Thirring model on the whole distance (energy) scale. We perform perturbative and renormalization group analyses to obtain the short-distance asymptotics, and numerically evaluate the long-distance behavior by using the form factor expansion. Our results illustrate the use of bosonization and conformal perturbation theory in the renormalization group analysis of a fermionic theory, and numerically confirm the validity of the form factor expansion in the case of the $SU(2)$-Thirring model.

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1. Introduction

Correlation functions in 2D integrable models have attracted much attention from experts in Quantum Field Theory (QFT). The possibility of their exploration on the whole length (energy) scale is of great importance. It gives a rare opportunity to probe non-perturbatively general principles of QFT. There is also a pragmatic reason for this interest. The past two decades have witnessed experimental work to identify and study quasi one-dimensional systems (for a review, see [1,2]). There were collective efforts of many physicists to apply integrable QFT to describe such physical systems [3]. For this purpose, a non-perturbative treatment of the correlation functions in integrable models seems to be valuable. It is worth noting that in recent years, angle resolved photoemission spectroscopy has matured into a powerful experimental method for probing the electronic Green's functions in quasi one-dimensional systems [4]. Hence two-point fermion correlators in integrable theories deserve special consideration.

In this paper we are studying Schwinger's function (Green's function in the Euclidean region)

$$\langle\, \Psi_\sigma(x)\, \bar\Psi_{\sigma'}(0)\,\rangle$$

in the $SU(2)$-Thirring model, which is described by the Euclidean action[1]

$$\mathcal{A} = \int d^2x \left\{ \sum_{\sigma=\uparrow,\downarrow} \bar\Psi_\sigma \gamma^\mu \partial_\mu \Psi_\sigma + \frac{\pi g}{8}\, \big(\bar\Psi\gamma_\mu\vec\tau\Psi\big)^2 \right\}. \tag{1.1}$$

Here Ψ_σ is a doublet of Dirac Fermi fields, and the Pauli matrices $\vec\tau = (\tau^1, \tau^2, \tau^3)$ act on the "colour" indices $\sigma = \uparrow, \downarrow$. The QFT (1.1) possesses a variety of interesting properties [5]. For instance, it is an asymptotically free theory (for $g > 0$) with unbroken chiral symmetry, and its mass scale M appears through dimensional transmutation. Also, it is a popular model for the interacting one-dimensional electron gas; as is known [6,7,8,9,10], (1.1) describes the scaling limit of the half-filled Hubbard chain,

$$\mathbf{H}_{Hub} = -\sum_{j=-\infty}^{+\infty} \left\{ \sum_{\sigma=\uparrow,\downarrow} \big(c^\dagger_{j,\sigma} c_{j+1,\sigma} + c^\dagger_{j+1,\sigma} c_{j,\sigma} \big) + U\, \Big(c^\dagger_{j,\uparrow} c_{j,\uparrow} - \frac{1}{2} \Big) \Big(c^\dagger_{j,\downarrow} c_{j,\downarrow} - \frac{1}{2} \Big) \right\}.$$

where $\{\, c^\dagger_{j,\sigma}\,, c_{j',\sigma'}\,\} = \delta_{\sigma\sigma'}\, \delta_{jj'}$. More precisely, if one sends the coupling constant $U \to +0$, the correlation length

$$R_c = \frac{\pi}{2\sqrt{U}}\, e^{\frac{2\pi}{U}}$$

[1] The definition of the coupling constant g is not conventional, but convenient for our purposes.

diverges and the correlation functions in the Hubbard chain at large lattice separations assume certain scaling forms. In particular, if $|j-j'| \gg 1$, the equal-time fermion correlator can be written as

$$\langle\, c_{j',\sigma'}\, c^{\dagger}_{j,\sigma}\, \rangle \to \delta_{\sigma'\sigma}\ \frac{\sin\left(\frac{\pi}{2}\,(j'-j)\right)}{\pi\,(j'-j)}\ \ F(\,|j'-j|/R_c\,)\,. \tag{1.2}$$

The scaling function F here is directly related to the field-theoretic correlation function:

$$\langle\, \Psi_{\sigma'}(x)\, \bar{\Psi}_{\sigma}(0)\, \rangle = \frac{\delta_{\sigma'\sigma}}{2\pi}\, \frac{\gamma_\mu x^\mu}{|x|^2}\, F(M|x|)\ . \tag{1.3}$$

Our analysis of the fermion correlator is based, on the one hand, on recently proposed expressions for the form factors of soliton-creating operators (or topologically charged fields) in the sine-Gordon model [11][2], and on the other hand, on a conformal perturbative analysis of two-point correlation functions involving such fields. The form factor expressions can be used to obtain the long-distance behavior of these two-point functions, whereas Conformal Perturbation Theory (CPT) gives their short-distance expansion [14]. The interest in some of these topological fields stems from their rôle in fermionic theories. For instance, it is well-known that the sine-Gordon model is equivalent to the massive Thirring model [15]. The components of the Thirring fermion field are then associated with soliton-creating operators of topological charge ± 1 and Lorenz spin $\pm\frac{1}{2}$, and correlators of these operators in the sine-Gordon model are related to fermion correlators in the massive Thirring model [16]. More interestingly, the sine-Gordon theory is closely related to a model which is an integrable deformation of (1.1) [5]. This "deformed" (or anisotropic) $SU(2)$-Thirring model exhibits the so-called spin-charge separation, which is translated by its representation in terms of two bosonic theories, one for the charge part, one for the spin part. The spin part of the fermion field corresponds to soliton-creating operators of topological charge ± 1 and Lorenz spin $\pm\frac{1}{4}$ in the sine-Gordon model, and its charge part is related to similar operators in a free massless bosonic theory.

Although form factor expansions and CPT are very effective tools for the study of, respectively, the long-distance and the short-distance asymptotics of Schwinger's functions [14,17,18], one usually gets into trouble when trying to compare both predictions in a region where they are expected to be accurate enough. Indeed, in general, one has the freedom

[2] Without taking normalization into consideration, some of such form factors were considered previously in Refs. [12,13]

of choosing the overall multiplicative normalization in the CPT expansion as well as in the form factor expansion, and there is no systematic way of relating both normalizations. For the case of the soliton-creating operators, the constant relating both normalizations was conjectured in [11]. It allows one to make unambiguous numerical predictions on the correlation functions of soliton-creating fields on the whole distance scale using the combined CPT and form factor data. We performed this calculation for the case of the $SU(2)$-Thirring fermion.

The paper is organized as follows. In Section 2, we recall some standard results concerning the anisotropic $SU(2)$-Thirring model and its relation to the sine-Gordon theory. In Section 3, the short-distance behavior of correlators of the soliton-creating operators is examined by means of CPT. Here we also perform a Renormalization Group (RG) resummation of the perturbative expansion in the vicinity of the Kosterlitz-Thouless point which corresponds to the $SU(2)$ limit of the fermion theory. In Section 4, the perturbative calculation is adapted to the momentum space fermion Schwinger's function; we give the two-point function in the $SU(2)$-Thirring model to third order in the running coupling. This particular result was recently obtained by standard perturbation theory in the modified Minimal Subtraction ($\overline{\text{MS}}$) scheme [19] (calculations in [19] concern, in fact, fermion correlators in a general non-abelian Thirring model). We then compare the result of Ref. [19] with ours and explicitly relate our RG scheme to the $\overline{\text{MS}}$ scheme. In Section 5, the long-distance behavior of the fermion correlator in the anisotropic $SU(2)$-Thirring model is analyzed by means of the form factors given in [11]. In Section 6, we examine properties of the fermion spectral density in the $SU(2)$-Thirring model. The outcome of our calculations is discussed in Section 7, where we numerically compare the short-distance behavior of the scaling function F (1.2), (1.3) (from the RG analysis) with its long-distance behavior (from its form factor expansion).

Finally, we note that this work has an essential overlap with Ref. [20], where (1.1) was considered as a model of one-dimensional Mott insulators.

2. Bosonization of the anisotropic $SU(2)$-Thirring model

The $SU(2)$-invariant Thirring model admits an integrable generalization such that the underlying $SU(2)$ symmetry is explicitly broken down to $U(1) \otimes \mathbb{Z}_2$:

$$\mathcal{A}_{DTM} = \int d^2x \left\{ \sum_{\sigma=\uparrow,\downarrow} \bar{\Psi}_\sigma \gamma_\mu \partial^\mu \Psi_\sigma + \frac{\pi g_\parallel}{8} \, J^3_\mu J^3_\mu + \frac{\pi g_\perp}{8} \left(J^1_\mu J^1_\mu + J^2_\mu J^2_\mu \right) \right\}, \tag{2.1}$$

where

$$J_\mu^A = \bar{\Psi}\gamma_\mu \tau^A \Psi \tag{2.2}$$

are vector currents. The model (2.1) is renormalizable, and its coupling constants $g_\parallel$, $g_\perp$ should be understood as "running" ones. In particular, in the RG-invariant domain $g_\parallel \geq |g_\perp|$, all RG trajectories originate from the line $g_\perp = 0$ of UV stable fixed points, and (2.1) indeed defines a quantum field theory[3]. Hence, in this domain (which is the only one that we discuss here), each RG trajectory is uniquely characterized by the limiting value

$$\rho = \frac{1}{2}\ \lim_{\ell\to 0}\, g_\parallel(\ell) \tag{2.3}$$

of the running coupling $g_\parallel(\ell)$ at extremely short distances (ℓ stands for the length scale), i.e. the theory (2.1) depends only on the dimensionless parameter ρ, besides the mass scale M appearing through dimensional transmutation.

As is well known (see e.g. [3,5]), the model (2.1) can be bosonized in terms of the sine-Gordon field $\varphi(x)$,

$$\mathcal{A}_{sG} = \int d^2x \left\{ \frac{1}{16\pi}\,(\partial_\nu\varphi)^2 - 2\mu\, \cos(\beta\varphi) \right\}, \tag{2.4}$$

with the coupling constant β in (2.4) related to ρ (2.3) by

$$\beta^2 = \frac{1}{1+\rho}\,, \tag{2.5}$$

and a free massless boson. Then the mass scale M is identified with the mass of the sine-Gordon solitons, which is related to the parameter μ by [21]

$$\mu = \frac{\Gamma(\frac{1}{1+\rho})}{\pi\Gamma(\frac{\rho}{1+\rho})}\ \left[M\ \frac{\sqrt{\pi}\Gamma(\frac{1}{2}+\frac{1}{2\rho})}{2\,\Gamma(\frac{1}{2\rho})}\right]^{\frac{2\rho}{1+\rho}}\ . \tag{2.6}$$

The precise operator relations between (2.1) and (2.3) can be found in [11]. In particular, for the two-point fermion correlator, the bosonization implies that

$$\langle\, \Psi_\sigma(x)\, \bar{\Psi}_{\sigma'}(0)\, \rangle = \frac{\delta_{\sigma'\sigma}}{2\pi}\ \frac{\gamma_\mu x^\mu}{|x|^{\frac{3}{2}}}\quad F^{(1)}_{1/4}(r)\ , \tag{2.7}$$

[3] The Hamiltonians corresponding to opposite choices of the sign of $g_\perp$ are unitary equivalent, so the sign of this coupling does not affect the physical observables.

where we use the notation $F^{(n)}_{\omega}$ $(n=1,\ \omega=1/4)$ for the real function which depends only on the distance $r=|x|$ (and implicitly on the the mass scale M and the parameter ρ), and which, in essence, coincides with the Euclidean correlator of nonlocal topologically charged fields in the model (2.4):

$$\langle\, \mathcal{O}^{n}_{-\omega\beta}(x)\, \mathcal{O}^{-n}_{\omega\beta}(0)\,\rangle = \Big(e^{i\pi}\, \frac{\bar{\mathrm{z}}}{\mathrm{z}} \Big)^{\omega n} \quad F^{(n)}_{\omega}(r)\,, \tag{2.8}$$

where $\mathrm{z}=\mathrm{x}^1+i\mathrm{x}^2$, $\bar{\mathrm{z}}=\mathrm{x}^1-i\mathrm{x}^2$. Again we refer the reader to the paper [11] for the precise definition of the field $\mathcal{O}^{n}_{a}$ $(a=\omega\beta)$. Here we note that it carries an integer topological charge n, a scale dimension

$$d=\frac{2\omega^2}{1+\rho}+\frac{n^2}{8}\,(1+\rho)\ , \tag{2.9}$$

and a Lorentz spin ωn.

3. Short-distance expansion

3.1. Conformal perturbation theory

We now turn to the analysis of the short-distance behavior of the correlator (2.8). In general, one can examine this behavior via the operator product expansion, for instance:

$$F^{(n)}_{\omega}(r)=\mathbb{C}_{\mathbf{I}}(r)+\mathbb{C}_{\cos(\beta\varphi)}(r)\ \langle\,\cos(\beta\varphi)\,\rangle+\dots\ . \tag{3.1}$$

The structure functions ($\mathbb{C}_{\mathbf{I}}(r)$, $\mathbb{C}_{\cos(\beta\varphi)}(r)$, etc.) admit power series expansions in μ^2, which can be obtained by using the standard rules of CPT, whereas the vacuum expectation values of the associated operators are in general non-analytical at $\mu=0$. In the perturbative treatment, we regard the sine-Gordon model (2.4) as a Gaussian conformal field theory

$$\mathcal{A}_{Gauss}=\int d^2x\,\frac{1}{16\pi}\,(\partial_\nu\varphi)^2 \tag{3.2}$$

perturbed by the relevant operator $\cos(\beta\varphi)$. Notice that in the limit $\mu\to 0$, the nonlocal topologically charged fields $\mathcal{O}^{n}_{a}$ can be expressed in terms of the right and left moving parts of a free massless field $\varphi=\varphi_R(\mathrm{z})-\varphi_L(\bar{\mathrm{z}})$ governed by the action (3.2):

$$\mathcal{O}^{n}_{a}\big|_{\mu\to 0}\to\tilde{\mathcal{O}}^{n}_{a}=\exp\Big\{\, i\,\Big(a-\frac{n}{4\beta}\Big)\,\varphi_R(\mathrm{z})-i\,\Big(a+\frac{n}{4\beta}\Big)\,\varphi_L(\bar{\mathrm{z}})\,\Big\}\ . \tag{3.3}$$

CPT gives the structure function $\mathbb{C}_{\mathbf{I}}$ (3.1) in the form

$$\left(e^{i\pi}\,\frac{\bar{\mathrm{z}}}{\mathrm{z}}\right)^{\omega n}\ \mathbb{C}_{\mathbf{I}}(r)=\Big\langle\,\tilde{\mathcal{O}}^{-n}_{\omega\beta}(x)\tilde{\mathcal{O}}^{n}_{-\omega\beta}(0)\exp\Big(2\mu\int' d^2y\,\cos(\beta\varphi)\Big)\,\Big\rangle_{Gauss}\,, \tag{3.4}$$

where $\langle\ldots\rangle_{Gauss}$ is the expectation value in the Gaussian theory $\mathcal{A}_{Gauss}$ and the exponential is understood as a perturbative series in μ. In the perturbative series, the integrals will have *power law* IR divergences which should be thrown away [14]. Such a regularization prescription is indicated by the prime near the integral symbol. In the absence of logarithmic divergences, throwing away the divergences is equivalent to treating the integrals as analytical continuations in the field dimensions [14]. Considering only the part of $F^{(n)}_{\omega}$ perturbative in μ, it is a simple matter to obtain

$$F^{(n)}_{\omega}(r)=r^{-2d}\left\{1+J_n(2\omega\beta^2,-2\beta^2)\ \mu^2\,r^{4-4\beta^2}+O\Big(r^{8-8\beta^2},r^2\Big)\right\}\,, \tag{3.5}$$

where d is given by (2.9) and

$$\begin{aligned}J_n(a,c)=\int' d^2xd^2y\ x^{a+\frac{n}{2}}\,\bar{x}^{a-\frac{n}{2}}\,(1-x)^{-a-\frac{n}{2}}(1-\bar{x})^{-a+\frac{n}{2}}\times\\ y^{-a-\frac{n}{2}}\,\bar{y}^{-a+\frac{n}{2}}\,(1-y)^{a+\frac{n}{2}}(1-\bar{y})^{a-\frac{n}{2}}\,|x-y|^{2c}\ .\end{aligned} \tag{3.6}$$

Two comments are in order here. First, the next omitted term in the short distance expansion (3.5) comes from either the next term in the perturbative series for $\mathbb{C}_{\mathbf{I}}$ $\big(O(r^{8-8\beta^2})\big)$ or from the leading contribution of $\cos(\beta\varphi)$ $\big(O(r^2)\big)$ in (3.1). Therefore, the μ^2 term written in (3.5) is a leading correction to the scale invariant part of the correlation function for $\frac{1}{2}<\beta^2<1$ only. Second, in writing (3.5) we specify the overall multiplicative normalization of the nonlocal topologically charged field $\mathcal{O}^{n}_{\omega\beta}$ by the condition

$$F^{(n)}_{\omega}(r)\to r^{-2d}\qquad \text{as}\quad r\to 0\ . \tag{3.7}$$

The integral (3.6) can be calculated using, for instance, techniques illustrated in [22]. The result can be expressed in terms of two generalized hypergeometric functions at unity:

$$\begin{aligned}A(q,c)&={}_3F_2(-c,-c-1,1-q;-c-q,2;1)\\ B(q,c)&={}_3F_2(q,q+1,c+2;c+q+2,c+q+3;1)\ .\end{aligned} \tag{3.8}$$

With $q=a+n/2$ and $\bar{q}=a-n/2$, we found:

$$J_n(a,c)=J^{(1)}+J^{(2)}+J^{(3)}+J^{(4)}\ , \tag{3.9}$$

where

$$
\begin{aligned}
J^{(1)} =& q\bar{q}\ \Gamma(1-q)\Gamma(1-\bar{q})\,\Gamma(1+c+q)\Gamma(1+c+\bar{q})\Gamma^2(-1-c)\times \\
& \big(\cos(\pi(q-\bar{q}))-\cos(\pi c)\cos(\pi(q+\bar{q}+c))\big)\ A(q,c)A(\bar{q},c)\,, \\
J^{(2)} =& \frac{\pi^2 q\Gamma(1+c)\Gamma(1+\bar{q})\Gamma(1+c+q)\Gamma(-1-c-\bar{q})}{\Gamma(q)\Gamma(-\bar{q})\Gamma(3+c+\bar{q})}\ A(q,c)B(\bar{q},c)\,, \\
J^{(3)} =& \frac{\pi^2 \bar{q}\Gamma(1+c)\Gamma(1+q)\Gamma(1+c+\bar{q})\Gamma(-1-c-q)}{\Gamma(\bar{q})\Gamma(-q)\Gamma(3+c+q)}\ B(q,c)A(\bar{q},c)\,, \\
J^{(4)} =& -\frac{\pi^2\Gamma^2(1+q)\Gamma^2(2+c)\Gamma(-1-c-\bar{q})\Gamma(-2-c-\bar{q})}{\Gamma^2(-\bar{q})\Gamma^2(-c)\Gamma(2+c+q)\Gamma(3+c+q)}\ B(q,c)B(\bar{q},c)\,.
\end{aligned}
$$

Notice that for $n=0$, the integral (3.6) was calculated previously in the work [23] (see also Ref. [24]).

3.2. Renormalization group resummation

Here we discuss the short-distance expansion of the correlator (2.8) for β^2 sufficiently close to unity. For this purpose, it is convenient to use the notation

$$
\epsilon = 1-\beta^2 \ll 1\ . \tag{3.10}
$$

Our previous CPT analysis suggests the following expansion for the structure function $\mathbb{C}_{\mathbf{I}}$:

$$
\mathbb{C}_{\mathbf{I}}(r) = r^{2d}\left\{1+\sum_{k=1}^{\infty} c_k\left(\mu r^{2\epsilon}\right)^{2k}\right\}\,, \tag{3.11}
$$

where the coefficients c_k are given by certain $4k$-fold Coulomb-type integrals. Evidently, this expansion cannot be directly applied in the limit $\epsilon\to 0$, where the perturbation $\cos(\beta\varphi)$ of the Gaussian action (3.2) becomes marginal. However, being expressed as a function of the scaling distance Mr, the structure function $\mathbb{C}_{\mathbf{I}}(r)$ should admit the following form:

$$
\mathbb{C}_{\mathbf{I}}(r) = \ Z_{n,\omega}\ \mathbb{C}_{\mathbf{I}}^{(ren)}(Mr)\ , \tag{3.12}
$$

where the r-independent renormalization constant $Z_{n,\omega}$ absorbs all divergences at $\epsilon=0$ and renders the renormalized structure function $\mathbb{C}_{\mathbf{I}}^{(ren)}$ finite in this limit. The divergences of the renormalization constant $Z_{n,\omega}$ should be directly related to the singularities of $\mathbb{C}_{\mathbf{I}}^{(ren)}$ at $Mr=0$; they point out that the power law asymptotic behavior (3.5) is modified by logarithmic corrections at $\epsilon=0$.

In order to explore the short-distance behavior for $\epsilon \ll 1$, it is convenient to return to the fermion description. Being essentially the corresponding structure function in the renormalizable QFT (2.1), $\mathbb{C}_{\mathbf{I}}(r)$ obeys the Callan-Symanzik equation. Therefore it can be written in the form:

$$\mathbb{C}_{\mathbf{I}}(r) = r^{-2d} \ \exp\Big\{ -2\int_0^r \frac{dr}{r}\left(\Gamma_g - d\right)\Big\} \ . \tag{3.13}$$

Here the function Γ_g is supposed to have a regular power series expansion in terms of the running coupling constants $g_{\parallel,\perp} = g_{\parallel,\perp}(r)$:

$$\Gamma_g = \sum_{l,k=0}^{\infty} \gamma_{lk} \ g_{\parallel}^{l} \, g_{\perp}^{2k} \ . \tag{3.14}$$

Notice that only even powers of the coupling $g_\perp$ appear in this expansion (see footnote #3). In writing (3.13), we use the normalization condition (3.7), and take into account that the UV limiting value of Γ_g coincides with the scale dimension (2.9),

$$\lim_{r\to 0} \Gamma_g = d \ . \tag{3.15}$$

We have also assumed that there is no resonance mixing of the operator $\mathcal{O}^{n}_{\omega\beta}$ with other fields, so it is renormalized as a singlet. One can easily check that this is indeed the case for the operators with $n = \pm 1$ and $|\omega| < 2$.

Condition (3.15) already encloses an important restriction on the series (3.14). Indeed, using Eqs. (2.3) and (2.9) along with the condition that the line of UV stable fixed points corresponds to $g_\perp = 0$, one obtains

$$\Gamma_g = \Gamma^{(0)}(g_\parallel) + \Gamma^{(1)}(g_\parallel)\, g_\perp^2 + \Gamma^{(2)}(g_\parallel)\, g_\perp^4 + O(g_\perp^6) \ , \tag{3.16}$$

where

$$\Gamma^{(0)}(g_\parallel) = \frac{2\omega^2}{1+\frac{g_\parallel}{2}} + \frac{n^2}{8}\left(1 + \frac{g_\parallel}{2}\right) .$$

The values of the other coefficients $\gamma_{l,k\geq 1}$ appearing in (3.14) essentially depend on the choice of a renormalization scheme, i.e on the precise specification of the running coupling constants. The latter obey the RG equations

$$\begin{aligned} r\,\frac{dg_\parallel}{dr} &= \frac{g_\perp^2}{f_\parallel(g_\parallel, g_\perp)} \\ r\,\frac{dg_\perp}{dr} &= \frac{g_\parallel g_\perp}{f_\perp(g_\parallel, g_\perp)} \ . \end{aligned} \tag{3.17}$$

Perturbatively, $f_{\|}(g_{\|}, g_{\perp})$ and $f_{\perp}(g_{\|}, g_{\perp})$ admit loop expansions as power series in $g_{\|}$ and $g_{\perp}$. In this work, we will use the scheme introduced by Al.B. Zamolodchickov [21,25]. He showed that under a suitable diffeomorphism in $g_{\|}$ and $g_{\perp}$, the functions $f_{\|}$ and $f_{\perp}$ can be chosen to be equal to each other, and furthermore, to be equal to

$$f_{\|} = f_{\perp} = 1 + \frac{g_{\|}}{2} \ . \tag{3.18}$$

With this choice for the β-function, the RG equations (3.17) can be integrated. To do this, we note that this system of differential equations has a first integral, the numerical value of which is determined through the condition (2.3),

$$g_{\|}^2 - g_{\perp}^2 = (2\rho)^2 \ . \tag{3.19}$$

Using (3.19), (3.10) and (2.5), equations (3.17) are solved as

$$g_{\|} = 2\rho \ \frac{1+q}{1-q} \ , \qquad g_{\perp} = \rho \ \frac{4\sqrt{q}}{1-q} \ , \tag{3.20}$$

where

$$q \left(\frac{1-q}{\rho}\right)^{-2\epsilon} = (r\Lambda)^{4\epsilon} \, . \tag{3.21}$$

The normalization scale Λ is another integration constant of the system (3.17). It is of the order of the physical mass scale and supposed to have a regular loop expansion,

$$\Lambda = M \ \exp\left(\tau_0 + \tau_1\rho + \tau_2\rho^2 + \ldots\right) \ . \tag{3.22}$$

It should be noted that the even coefficients τ_0, $\tau_2, \ldots$ are essentially ambiguous and can be chosen at will. A variation of these coefficients corresponds to a smooth redefinition of the coupling constants which does not affect the β-function. By contrast, the odd constants τ_{2k+1} are unambiguous and precisely specified once the form of the RG equations is fixed. It is possible to show [21,25] that the odd constants vanish in the Zamolodchikov's scheme:

$$\tau_{2k+1} = 0 \qquad (k = 0, 1 \ldots) \ .$$

Once the coefficients τ_{2k} in (3.22) are chosen, the running coupling constants are completely specified, and all coefficients in the power series expansion (3.14) are determined unambiguously. They can be explicitly calculated by comparing the CPT result (3.5) with the form (3.13). From (3.13),

$$\Gamma_g = -\frac{1}{2} \ r\,\partial_r \log\left(\mathbb{C}_{\mathbf{I}}\right)$$

and, as it follows from the general CPT expansion (3.11) and the definition (3.21) of q, the function Γ_g can be expanded in powers of q. Explicitly, using the CPT result (3.5),

$$\Gamma_g = d - 2\,\epsilon\,\left(\frac{\sqrt{\rho}}{\Lambda}\right)^{4\epsilon}\mu^2\; J_n\big(2\omega(1-\epsilon), 2\epsilon - 2\big)\; q + O(q^2)\;. \tag{3.23}$$

Moreover, the coefficients in this expansion are power series in ρ. For example, using Eqs. (2.6) and (3.22), it is easy to show that

$$\frac{\pi\mu}{\epsilon}\left(\frac{\sqrt{\rho}}{\Lambda}\right)^{2\epsilon} = \exp\left\{ -2\bar{\tau}_0\rho + \Big(2\bar{\tau}_0 - \frac{1}{2}\Big)\rho^2 - \Big(2\tau_2 + 2\bar{\tau}_0 - \frac{2}{3}\,\zeta(3) - \frac{1}{2}\Big)\,\rho^3 + O(\rho^4)\right\}. \tag{3.24}$$

Here and after, we set for convenience

$$e^{\tau_0} = \sqrt{\frac{\pi}{8}}\;\; e^{\gamma_E + \bar{\tau}_0}\;, \tag{3.25}$$

where $\gamma_E = 0.5772\ldots$ is the Euler constant. The integral $J_n\big(2\omega(1-\epsilon), 2\epsilon-2\big)$ appearing in (3.23) can also be expanded in powers of ρ, using $\epsilon = \rho/(1+\rho)$. In Appendix A, we quote the first few terms in the expansion of $J_n(a,c)$ (3.6) around $c = -2$, which are obtained through the use of (3.9). From this expansion, it is easy to obtain the expansion of $J_n\big(2\omega(1-\epsilon), 2\epsilon-2\big)$ in powers ρ. Then, one can compare the CPT expansion of Γ_g in q and ρ (3.23) with the corresponding expansion (3.14) coming from the RG analysis (where of course one should expand $g_\parallel$ and $g_\perp^2$ in q and ρ from (3.20)). This determines the coefficients $\gamma_{l,1}$ for $l = 0, 1, 2$. If we want an expression valid to order g^4, we need one more coefficient: $\gamma_{0,2}$. In principle, it can be obtained from the expansion in ρ of the coefficients c_2 in the series (3.11). In Section 5, we describe a way to find $\gamma_{0,2}$ without the cumbersome calculation beyond the lowest CPT order.

In order to simplify the form of the structure function (3.13), it is convenient, instead of using the coefficients $\gamma_{l,k}$, to parametrize the first few terms of the power series expansions $\Gamma^{(1,2)}(g_\parallel)$ (3.16) as:

$$\begin{aligned} \Gamma^{(1)}(g_\parallel) &= -\frac{1}{1+\frac{g_\parallel}{2}}\;\left\{\frac{n^2}{32} - \frac{u_1}{2} + v_1\, g_\parallel + \Big(v_2 - \frac{3u_2}{2}\Big)\, g_\parallel^2 + O(g_\parallel^3)\right\}\,, \\ \Gamma^{(2)}(g_\parallel) &= -\frac{v_2}{2} + O(g_\parallel)\;. \end{aligned} \tag{3.26}$$

The explicit values of the coefficients u_1, u_2, v_1 and v_2 in (3.26) are given in Appendix B.

Let us substitute (3.16) and (3.26) into Eq. (3.13). The RG flow equations (3.17) allow one to evaluate the integral and to write the structure function in the form (3.12) with

$$\begin{aligned} \mathbb{C}_{\mathbf{I}}^{(ren)} =& (Mr)^{-4\omega^2-n^2(1+\rho^2)/4} \ \left(g_\perp^2\right)^{\omega^2-n^2(1-\rho^2)/16} \times \\ & e^{-u_1 g_\parallel - u_2 g_\parallel^3} \ \Big(1+g_\perp^2(v_1+v_2 g_\parallel)+O(g^4)\Big) \ , \end{aligned} \tag{3.27}$$

and

$$Z_{n,\omega} = M^{2d} \ \left(2^{\rho+1}\sqrt{\rho}\, e^{\tau_0\rho+\tau_2\rho^3+\ldots}\right)^{n^2/2-2d} \quad e^{2\rho u_1+(2\rho)^3 u_2+\ldots} \ . \tag{3.28}$$

Notice that the transformation

$$Z_{n,\omega} \to e^{w_0+w_1(2\rho)^2+w_2(2\rho)^4+\ldots} \ \ Z_{n,\omega} \ , \tag{3.29}$$

where the series contains only even powers of ρ with arbitrary coefficients w_k, accompanied by the transformation

$$\mathbb{C}_{\mathbf{I}}^{(ren)} \to e^{-w_0-w_1(g_\parallel^2-g_\perp^2)-w_2(g_\parallel^2-g_\perp^2)^2+\ldots} \ \ \mathbb{C}_{\mathbf{I}}^{(ren)}$$

does not affect the structure function $\mathbb{C}_{\mathbf{I}}$ (3.12) due to relation (3.19).

Our prime interest in this work is the correlation function (2.7). For $n=1$ and $\omega=\frac{1}{4}$, the relations obtained above lead to the following perturbative expansion for the two-point fermion correlator in the anisotropic $SU(2)$-Thirring model:

$$\begin{aligned} \langle\, \Psi_{\sigma'}(x)\, \bar{\Psi}_\sigma(0)\,\rangle =& \frac{Z_\Psi \delta_{\sigma'\sigma}}{2\pi} \ \ \frac{\gamma_\mu x^\mu}{|x|^{2+\frac{\rho^2}{4}}} \ \ \left(g_\perp^2\right)^{\frac{\rho^2}{16}} \ \exp\left\{-\frac{3}{16}\, g_\parallel - \frac{\bar{\tau}_0}{32}\, g_\parallel^3\right\}\times \\ & \exp\left\{\frac{3}{16}\left(\bar{\tau}_0-\frac{1}{4}\right) g_\perp^2 - \frac{3}{16}\left(\bar{\tau}_0^2-\frac{1}{6}\,\bar{\tau}_0-\frac{1}{16}\right) g_\parallel g_\perp^2 + O(g^4)\right\} \ , \end{aligned} \tag{3.30}$$

where

$$Z_\Psi = (4\rho)^{-\frac{\rho^2}{8(1+\rho)}} \ \left(M\sqrt{\frac{\pi}{2}}\right)^{-\frac{\rho^3}{4(1+\rho)}} \ \exp\left\{\frac{3\rho}{8}-\frac{\gamma_E}{4}\,\rho^3+O(\rho^4)\right\} \ .$$

In Eq. (3.30), we use the notation $\bar{\tau}_0$ defined by (3.25).

We now set $\rho=0$ and $g_\parallel=g_\perp=g$ in (3.30) to obtain the perturbative expansion of the scaling function F (1.3) for the $SU(2)$-Thirring model,

$$F^{(pert)} = \exp\left\{-\frac{3}{16}\, g + \frac{3}{16}\left(\bar{\tau}_0-\frac{1}{4}\right) g^2 - \frac{3}{16}\left(\bar{\tau}_0^2-\frac{1}{16}\right) g^3 + O(g^4)\right\} \ . \tag{3.31}$$

Here the running coupling constant g solves the equation

$$-g^{-1} + \frac{1}{2}\,\ln(g) = \ln\Big(\sqrt{\frac{\pi}{2}}\,e^{\gamma_E+\bar{\tau}_0}\,Mr\Big)\ , \tag{3.32}$$

which is the limit $\rho = 0$ of Eqs. (3.20) and (3.21).

Let us stress here that, if the perturbation series could be summed, then the function F should not depend on the auxiliary parameter $\bar{\tau}_0$:

$$\frac{\partial F}{\partial \bar{\tau}_0} = 0\ .$$

This is, however, not true if we truncate the series (3.31) at some order N (for instance, if one leaves only the terms explicitly written in (3.31)). In this case,

$$\frac{\partial}{\partial \bar{\tau}_0}\,F_N^{(pert)} = O\big(g^{N+1}\big)\ ,$$

where the truncated series is denoted by $F_N^{(pert)}$. In fitting numerical data with (3.31), we may treat $\bar{\tau}_0$ as an optimization parameter, allowing us to minimize or at least develop a feeling for the effects of the remainder of the series. Similar ideas have been discussed for QCD in Ref. [26].

It may be worth mentioning that Eq.(3.27), along with explicit values of the coefficients quoted in Appendix B, allows one to immediately determine the short-distance expansion of some other conventional correlators in the (anisotropic) $SU(2)$-Thirring model. For example, since the sine-Gordon field φ (2.4) itself can be defined by the relation

$$\varphi = -i\,\frac{\partial}{\partial a}\mathcal{O}_a^n\Big|_{\substack{n=0\\ a=0}}\ ,$$

and the spin current J_μ^3 (2.2) is bosonized as

$$J_\mu^3 = \frac{\beta}{2\pi}\,\partial_\mu\varphi\ , \tag{3.33}$$

we can use (3.27) to obtain the short-distance expansion of the current-current correlator. For the $SU(2)$-Thirring model (1.1) one has,

$$\langle\, J_\mu^A(x)\, J_\nu^B(0)\,\rangle = \frac{\delta^{AB}}{\pi^2\,|x|^2}\ \Big\{\Big(\delta_{\mu\nu} - \frac{2x_\mu x_\nu}{|x|^2}\Big)\ I_1 + \delta_{\mu\nu}\ I_2\Big\}\ , \tag{3.34}$$

where

$$I_1 = 1 - \frac{g}{2} + \Big(\bar{\tau}_0 + \frac{1}{4}\Big)\frac{g^2}{2} - \bar{\tau}_0(\bar{\tau}_0+1)\,\frac{g^3}{2} + \Big(\frac{\bar{\tau}_0^3}{2} + \bar{\tau}_0^2 + \frac{\bar{\tau}_0}{4} - \frac{13}{128} - \frac{7}{16}\,\zeta(3)\Big)\,g^4 + O(g^5),$$
$$I_2 = \frac{g^2}{4}\,\Big\{1 - 2\,\bar{\tau}_0\,g + \bar{\tau}_0(3\,\bar{\tau}_0+1)\,g^2 - \Big(4\,\bar{\tau}_0^3 + \frac{7}{2}\,\bar{\tau}_0^2 + \frac{\bar{\tau}_0}{2} - \frac{13}{16} - \frac{7}{2}\,\zeta(3)\Big)\,g^3 + O(g^4)\Big\}\,,$$

and the running coupling constant g is the same as in (3.32).

4. Perturbative expansion of the momentum-space correlation function

4.1. Large-momentum asymptotics

Perturbative calculations of fermion Green's functions in renormalizable 2D models with four-fermion interaction are widely covered in the literature. The results in this domain are usually expressed in momentum space. Hence it seems appropriate at this point to adapt the calculation of the previous section to the momentum-space fermion correlator, giving a large-momentum expansion.

The RG analysis performed in the previous section can be applied in essentially the same way to the Fourier transform of the fermion correlator (2.7):

$$\int d^2x\, e^{-ipx} \left\langle \Psi_\sigma(x)\, \bar\Psi_{\sigma'}(0) \right\rangle = -i\, \delta_{\sigma\sigma'}\, \frac{\gamma^\mu p_\mu}{p^2}\ \tilde F(p^2)\ . \tag{4.1}$$

Here and after we use the notation $p^2 = p^\mu p_\mu$. From the result of CPT, (3.5), one immediately obtains the large momentum expansion of this Fourier transform:

$$\tilde F = Q(d_\Psi)\,(p^2)^{d_\Psi-\frac12}\left\{1+\frac{Q(d_\Psi-2\epsilon)}{Q(d_\Psi)}\,J_1(\beta^2/2,-2\beta^2)\ \mu^2(p^2)^{-2\epsilon}+O\big((p^2)^{-4\epsilon},p^{-2}\big)\right\}, \tag{4.2}$$

where

$$Q(a) = 2^{1-2a}\ \frac{\Gamma(\frac32-a)}{\Gamma(\frac12+a)}\ ,$$

and

$$d_\Psi = \frac12 + \frac{\rho^2}{4(1+\rho)}$$

is the scale dimension of the fermion field. The factor $Q(d_\Psi-2\epsilon)/Q(d_\Psi)$ is essentially the only source of differences between the RG treatments in coordinate space and in momentum space. The RG analysis in momentum space goes as in the previous section. The perturbative part in μ of $\tilde F$ obeys the Callan-Symanzik equation, so it can be written as

$$\tilde F^{(pert)} = Q(d_\Psi)\ (p^2)^{d_\Psi-\frac12}\ \exp\left\{-\int_{p^2}^{\infty}\frac{ds}{s}\,\big(\tilde\Gamma_g-d_\Psi\big)\right\}\ , \tag{4.3}$$

where the function $\tilde\Gamma_g$ admits a power series expansion in terms of the momentum-space running coupling constants $g_{\|,\perp} = g_{\|,\perp}(p^2)$ depending on the Lorentz invariant p^2:

$$\tilde\Gamma_g = \sum_{l,k=0}^{\infty}\tilde\gamma_{l,k}\ g_\|^l g_\perp^{2k}\ . \tag{4.4}$$

Notice that, with some abuse of notations, we use here the same symbols $g_{\|,\perp}$ for the momentum-space running couplings as we used for the coordinate-space running couplings. In order to fix the coefficients in (4.4), we have to choose a renormalization scheme. Substituting r by $1/\sqrt{p^2}$ in (3.21) defines Zamolodchikov's scheme in momentum space. It is a simple matter to repeat the steps of the previous section in order to determine the first few coefficients $\tilde{\gamma}_{l,1}$ in (4.4). Just compare the logarithmic derivatives of the expressions (4.2) and (4.3); the only difference is that the factor $Q(d_\Psi - 2\epsilon)/Q(d_\Psi)$ in (4.2) will have to be expanded in ρ, giving non-trivial contributions. As for the coefficients $\tilde{\gamma}_{l,2}$, one would in principle need the next order in CPT. However, again as in the previous section, it is possible to determine $\tilde{\gamma}_{0,2}$ without this calculation, as described in the next section. From these coefficients, and from the form of the RG flow equation, one can evaluate the integral in (4.3) and obtain the asymptotic behavior of the two-point function in the Euclidean region at $p^2 \to +\infty$. We quote here the result in the case of the $SU(2)$-Thirring model,

$$\tilde{F}^{(pert)} = \exp\left\{ -\frac{3}{16}\,g + \frac{3}{16}\left(\tilde{\tau}_0 - \frac{1}{4}\right) g^2 - \frac{3}{16}\left(\tilde{\tau}_0^2 - \frac{1}{16}\right) g^3 + O(g^4) \right\} . \tag{4.5}$$

Here

$$-g^{-1} + \frac{1}{2}\,\ln(g) = \ln\left(\sqrt{2\pi} M\, e^{\tilde{\tau}_0}/\sqrt{p^2}\,\right) , \tag{4.6}$$

and $\tilde{\tau}_0$ is an arbitrary parameter which can be chosen at will. Notice the strong similarity between (4.5) and (3.31).

We also quote here the corresponding function $\tilde{\Gamma}_g$ (4.4) in the case $g_\| = g_\perp$:

$$\tilde{\Gamma}_g = \frac{1}{2} + \frac{3}{32}\,g^2 - \frac{3}{16}\,\tilde{\tau}_0\; g^3 + \frac{3}{32}\left(3\,\tilde{\tau}_0^2 + \tilde{\tau}_0 - \frac{3}{16}\right) g^4 + O(g^5)\ . \tag{4.7}$$

4.2. Comparison with the four-loop conventional perturbation calculations

In [19], the anomalous dimension for the fermion field in the $\overline{\mathrm{MS}}$ scheme was found to fourth order for a general non-abelian Thirring model (see also [27] and references therein for a discussion of various aspects of dimensional regularization in the non-abelian Thirring model and for results to lower order). In contrast, we have calculated, in coordinate space, the two-point functions of more general operators, including the fermion fields, in the particular case of the $SU(2)$-Thirring model (and an anisotropic deformation of it), and we have sketched the equivalent calculation in momentum space for the fermion fields. We would now like to compare Eq.(4.5) with the $SU(2)$ case of the Ali-Gracey result [19]. In order to perform the comparison, we need to find the relation between our running coupling constant g and theirs, which will be denoted $g_{AG} = -\lambda$ [4], and then find the relation between our function $\tilde{\Gamma}_g$ (4.3) and their anomalous dimension, which we will denote γ_λ.

The coupling λ corresponds to the $\overline{\rm MS}$ scheme; the associated β-function was found in [27] to fourth order:

$$2\,p^2\,\frac{d\lambda}{dp^2} = \beta_\lambda = -\frac{\lambda^2}{\pi} + \frac{\lambda^3}{2\pi^2} - \frac{83}{128\pi^3}\;\lambda^4 + O(\lambda^5)\ . \tag{4.8}$$

By comparison, in the scheme that we use, the β-function (3.17), (3.18) is

$$2\,p^2\,\frac{dg}{dp^2} = \beta_g = -\frac{g^2}{1+g/2} = -g^2 + \frac{g^3}{2} - \frac{g^4}{4} + O(g^5)\ . \tag{4.9}$$

The difference in the factor multiplying the square of the coupling in these two expressions results only from a different normalization of the coupling in the action (see Eq.(1.1)). The relation between the couplings g and λ that corresponds to these different β-functions is

$$\frac{\lambda}{\pi} = g - \tau\,g^2 + \Big(\tau^2 + \frac{\tau}{2} + \frac{51}{128}\Big)\,g^3 + O(g^4)\ . \tag{4.10}$$

Here τ is some numerical factor which cannot be determined by comparing the β-functions: its variation modifies the choice of the normalization scale and doesn't affect the β-beta functions. The normalization scale for the $\overline{\rm MS}$ scheme is defined by imposing the following condition on the subleading asymptotics of the solution of the RG flow equation (4.8):

$$\frac{\lambda}{\pi} = \frac{1}{\ln\big(\sqrt{p^2}/\Lambda_{\overline{\rm MS}}\big)} + \frac{1}{2}\,\frac{\ln\ln(\sqrt{p^2}/\Lambda_{\overline{\rm MS}})}{\ln^2(\sqrt{p^2}/\Lambda_{\overline{\rm MS}})} + O\bigg(\frac{\ln^2\ln(\sqrt{p^2}/\Lambda_{\overline{\rm MS}})}{\ln^3(\sqrt{p^2}/\Lambda_{\overline{\rm MS}})}\bigg)\ . \tag{4.11}$$

(This implies that the term $O\big(1/\ln^2(\sqrt{p^2}/\Lambda_{\overline{\rm MS}})\big)$ does not appear in the expansion of λ.) From (4.6), (4.10) and (4.11), we find that

$$\Lambda_{\overline{\rm MS}} = \sqrt{2\pi}M\;e^{\tilde{\tau}_0-\tau}\ . \tag{4.12}$$

In [19], the perturbative part of the function $\tilde{F}$ (4.1) was calculated up to the overall multiplicative normalization to third order in λ. The result can be written in the following form

$$\tilde{F}^{(pert)} \propto \frac{1}{h_\lambda}\;\exp\bigg\{-\frac{1}{2}\int^{p^2}\frac{ds}{s}\;\gamma_\lambda\bigg\}\ ,$$

---

[4] Notice that in [19], the coupling constant g_{AG} is assumed to be negative, so $\lambda > 0$, which agrees with the sign of our coupling constant g.

where the function h_λ and the anomalous dimension γ_λ were given in [19] to fourth order in λ for the Thirring model with a general non-abelian symmetry. In the particular case of the $SU(2)$-symmetry, they specialize to

$$h_\lambda = 1 + \frac{15}{128\pi^2}\,\lambda^2 - \frac{11}{512\pi^3}\,\lambda^3 + \frac{3(80\zeta(3)-511)}{32768\,\pi^4}\,\lambda^4 + O(\lambda^5)\,, \tag{4.13}$$

and

$$\gamma_\lambda = -\frac{3}{16\pi^2}\,\lambda^2 + \frac{15}{64\pi^3}\,\lambda^3 + \frac{3}{1024\pi^4}\,\lambda^4 + O(\lambda^5)\ . \tag{4.14}$$

Comparing (4.3) in the case $\rho = 0$ with the above expressions, one has the following relation:

$$\tilde{\Gamma}_g = \frac{1}{2} - \frac{\gamma_\lambda}{2} - \frac{\beta_\lambda}{2}\,\frac{d}{d\lambda}\,\log(h_\lambda)\ . \tag{4.15}$$

Using Eqs. (4.10)-(4.14), one can check that our result (4.7) agrees with (4.15), provided that

$$\tau = \tilde{\tau}_0\ . \tag{4.16}$$

Notice that the relation between the normalization scale $\Lambda_{\overline{\rm MS}}$ and M,

$$\Lambda_{\overline{\rm MS}} = \sqrt{2\pi}M\ , \tag{4.17}$$

which is a consequence of (4.12) and (4.16), was previously found in Ref. [10].

5. Long-distance behavior

Here we concentrate on the long-distance behavior of Schwinger's function (2.8) for $n = 1$ and $\omega = 1/4$. Let us recall that for $\frac{1}{2} < \beta^2 \le 1$, there are only solitons and antisolitons in the spectrum of the sine-Gordon model. We will denote them by A_- and A_+ respectively. The conservation of the topological charge,

$$\frac{\beta}{2\pi}\int_{-\infty}^{\infty} d{\rm x}\,\partial_{\rm x}\varphi\ ,$$

implies that the non-vanishing form factors of the operator $\mathcal{O}^{+1}_{\beta/4}$ are of the form

$$\langle\, vac\mid \mathcal{O}^{+1}_{\beta/4}(0)\mid A_-(\theta_1)\cdots A_-(\theta_{N+1})\,A_+(\theta'_1)\cdots A_+(\theta'_N)\,\rangle\ , \tag{5.1}$$

where θ_i and θ'_j denote rapidities of solitons and antisolitons respectively. Up to an overall normalization, all these form factors can be written down in closed form, as certain N-fold integrals [28,29,12]. The spectral decomposition for the correlation function (2.7) then gives

$$
\begin{aligned}
F^{(1)}_{1/4}(r) = \int_{-\infty}^{+\infty}\frac{d\theta}{2\pi}\; e^{-Mr\cosh(\theta)}\,|\langle\, vac \mid \mathcal{O}^{+1}_{\beta/4}(0)\mid A_-(\theta)\,\rangle|^2 + \frac{1}{3!}\int_{-\infty}^{\infty}\frac{d\theta_1 d\theta_2 d\theta_3}{(2\pi)^3}\times \\
e^{-Mr\sum_{k=1}^3\cosh(\theta_k)} \sum_{\sigma_1+\sigma_2+\sigma_3=-1} |\langle\, vac \mid\; \mathcal{O}^{+1}_{\beta/4}(0)\mid A_{\sigma_1}(\theta_1)A_{\sigma_2}(\theta_2)A_{\sigma_3}(\theta_3)\,\rangle|^2 + \ldots\ ,
\end{aligned}
\tag{5.2}
$$

where the dots stand for the five-particle and higher contributions, which are of the order of e^{-5Mr}. The long-distance asymptotic behavior of the correlation function is dominated by the contribution of the one-particle states,

$$
\langle\, vac \mid \mathcal{O}^{+1}_{\beta/4}(0)\mid A_-(\theta)\,\rangle = \sqrt{\mathbf{Z}_1(\beta/4)}\; e^{\frac{1}{4}\,(\theta+\frac{i\pi}{2})}\ ,
$$

and has an especially simple form,

$$
F^{(1)}_{1/4}(r) = \mathbf{Z}_1(\beta/4)\,\Big\{\,\frac{e^{-Mr}}{\sqrt{2\pi Mr}} + O(e^{-3Mr})\,\Big\}\ . \tag{5.3}
$$

Here we use the notation $\mathbf{Z}_n(a)$ $(a=\omega\beta)$ from work [11] for the field-strength renormalization which controls the long-distance asymptotics of the correlation function (2.8). Let us stress here that the overall multiplicative normalization of the field $\mathcal{O}^1_{\beta/4}$ was already fixed by the condition (3.7), hence the constant $\mathbf{Z}_1(\beta/4)$ is totally unambiguous. In [11], the following explicit formula for $\mathbf{Z}_n(\omega\beta)$ was proposed:

$$
\begin{aligned}
\mathbf{Z}_n(\omega\beta) = \Big(\frac{\mathcal{C}_2}{2\,\mathcal{C}_1^2}\Big)^{\frac{n}{2}}\;\Big(\frac{\mathcal{C}_2}{16\rho}\Big)^{-\frac{n^2}{4}}\;\Bigg[\frac{\sqrt{\pi}M\Gamma(\frac{3}{2}+\frac{1}{2\rho})}{2\,\Gamma(1+\frac{1}{2\rho})}\Bigg]^{2d}\times \\
\exp\Bigg[\int_0^\infty\frac{dt}{t}\,\Big\{\frac{\cosh(4\omega t)\,e^{-(1+\rho)nt}-1}{2\,\sinh(t)\sinh\big((1+\rho)t\big)\,\cosh(t\rho)} + \frac{n}{2\,\sinh(t)} - 2d\;e^{-2t}\Big\}\Bigg]\ .
\end{aligned}
\tag{5.4}
$$

In this formula, d is the scale dimension (2.9) and the constants $\mathcal{C}_1,\ \mathcal{C}_2$ read

$$
\begin{aligned}
\mathcal{C}_1 &= \frac{2^{-\frac{5}{12}}\,e^{\frac{1}{4}}\,\Gamma(\frac{1}{4})}{\sqrt{\pi}\,A_G^3}\ \exp\Big\{\int_0^\infty\frac{dt}{t}\ \frac{\sinh^2(\frac{t\rho}{2})\,e^{-t}}{2\,\cosh^2(t\rho)\,\sinh(t)}\Big\}, \\
\mathcal{C}_2 &= \frac{\Gamma^4(\frac{1}{4})}{4\pi^3}\ \exp\Big\{-2\int_0^\infty\frac{dt}{t}\ \frac{\sinh^2(\frac{t\rho}{2})\,e^{-t}}{\cosh(t\rho)\,\sinh(t)}\Big\}\ ,
\end{aligned}
$$

where $A_G = 1.282427\ldots$ is the Glaisher constant.

We do not write down explicitly the general formula for the three-particle contribution in (5.3) because it is a rather mechanical substitution of relations presented in [11]. (For

$\beta^2 = 1$ the corresponding formulas can be found in Appendix C.) Here we make the following observation concerning the $\beta^2 \to 1$ limit. The examination of (5.2) based on explicit formulas for the form factors shows that the function $\big[\mathbf{Z}_1(\beta/4)\big]^{-1} F^{(1)}_{1/4}$ admits an asymptotic power series expansion in terms of the variable ρ^2. In other words, all divergences at $\rho^2 \to 0$ of $F^{(1)}_{1/4}$, considered as a function of the variables ρ^2 and Mr, are absorbed by the normalization constant $\mathbf{Z}_1(\beta/4)$. Using Eq. (5.4), one can check that the constant $\mathbf{Z}_1(\beta/4)$ admits exactly the same type of singular behavior at $\rho^2 = 0$ as the constant $Z_{n,\omega}$ (3.28) for $n = 1$, $\omega = 1/4$, and also that

$$\frac{\mathbf{Z}_1(\beta/4)}{Z_{1,1/4}} = 2^{-\frac{1}{3}}\ \sqrt{\pi}\, e^{-\frac{1}{4}}\ A_G^3\ \exp\big(w_1\,\rho^2 + O(\rho^3)\big)\ . \tag{5.5}$$

The explicit form of the coefficient w_1 is not essential here. What is important is that the linear term in ρ does not appear in the expansion (5.5). This observation can be immediately generalized and checked for any n and ω. Furthermore, we expect that

$$\log\Big(\frac{\mathbf{Z}_n(\omega\beta)}{Z_{n,\omega}}\Big) = \sum_{k=0}^{\infty} w_k\ \rho^{2k} + O(\rho^\infty)\ , \tag{5.6}$$

where $\mathbf{Z}_n(\omega\beta)$ is the normalization constant (5.4). In other words, by means of the transformation (3.29) with properly chosen coefficients w_k, the constant $Z_{n,\omega}$ in (3.12) can be set to be equal (in a sense of formal power series) to $\mathbf{Z}_n(\omega\beta)$. At the moment, we do not have a rigorous proof of (5.6). But it leads to some interesting prediction to be checked. As was already mentioned, the calculations performed in the leading CPT order determine only the combination $v_2 - 3u_2/2$, but do not fix the individual values of the coefficients u_2 and v_2 in the series (3.27). Accepting (5.6), one can immediately find the values of the coefficients u_2 (see Appendix B). In the case $n = 1$, $\omega = \frac{1}{4}$, it allows one to extend the perturbative expansion (3.31), as well as the equivalent expansion (4.5), to order g^3. As was discussed in Section 4, Eq. (4.5) is in a complete agreement with the result of four-loop perturbative calculations from [19]. This in fact shows that the ρ^3-term really is absent in the series (5.5).

6. Spectral density

The spectral density is an important quantity related to the two-point function and its analytical structure in momentum space. It is often what is measured in actual condensed matter experiments [4,20], and it allows one to completely reconstruct the two-point function. In this section, we discuss the properties of the spectral density in the $SU(2)$-Thirring model.

The spectral decomposition of the fermion Green's function yields the following form for the function $\tilde{F}$ (4.1):

$$\tilde{F}(p^2) = 1 - \int_{M^2}^{+\infty} ds\, \frac{\Delta\tilde{F}(s)}{p^2+s}\ . \tag{6.1}$$

The notation $\Delta\tilde{F}$ for the spectral density reminds us that $\tilde{F}$, considered as a function of one complex variable p^2, has a branch cut in the Minkowski region $p^2 < 0$ starting at $p^2 = -M^2$, and that the spectral density can be recovered from the discontinuity along this cut:

$$\Delta\tilde{F}(s) = \frac{1}{2\pi i}\ \big(\, \tilde{F}(e^{i\pi}s) - \tilde{F}(e^{-i\pi}s)\,\big)\ . \tag{6.2}$$

The easiest way to obtain the large s asymptotics of the spectral density is to use the expansion (4.5) along with knowledge of the analytical properties of the coupling constant g (4.6) as a function of the complex variable p^2. Notice that g can be expressed in terms of the principal branch of the product log (or Lambert) function, which gives the solution for W in $W\,e^W = z$ (see e.g. [30]):

$$g = 2\ W^{-1}\Big(\frac{p^2 e^{-2\tilde{\tau}_0}}{\pi M^2}\Big)\ . \tag{6.3}$$

The principal branch of the W-function analytically maps the complex z-plane minus the branch cut $z \in]-\infty, -e^{-1}]$ to the part of the complex W-plane enclosing the real axis and delimited by the curve $\Re e\, W = -\Im m\, W\ \cot(\Im m\, W)$ for $-\pi < \Im m\, W < \pi$. The analyticity implies that the power series

$$\sum_{n=0}^{\infty} \frac{1}{n!}\ \Big(i\phi\, z\, \frac{d}{dz}\Big)^n\ W(z)\Big|_{z=s}$$

converges for real positive $s > e^{-1}$ and $|\phi| \leq \pi$ and coincides with $W(e^{i\phi}s)$. Similar considerations are, of course, valid for the coupling constant g (6.3). In particular, for sufficiently large s,

$$g(e^{\pm i\pi}s) = \sum_{n=0}^{\infty} \frac{1}{n!}\ \Big(\pm\, i\pi\, p^2\, \frac{d}{dp^2}\Big)^n\ g(p^2)\Big|_{p^2=s}\ .$$

This then gives us, with (4.5) and the RG flow equation (4.9), the asymptotic expansion of the spectral density for large s. It can be written in the following form:

$$\Delta\tilde{F}(s) = -\frac{g^2}{2}\left\{1 - \frac{g}{2} - \frac{\pi^2 - 1}{4}\, g^2 + O(g^3)\right\} \frac{\partial \tilde{F}^{(pert)}}{\partial g}\Big|_{p^2=s} . \tag{6.4}$$

Here the function $\tilde{F}^{(pert)}$ is given by (4.5) and g is defined by the equation (4.6).

Now let us consider the threshold behavior of the spectral density. According to the analyses of the previous section, the long-distance asymptotic behavior of the scaling function F (1.3) is described by the expansion

$$F = F^{(1)} + F^{(3)} + O(e^{-5Mr}) , \tag{6.5}$$

where

$$F^{(1)} = C\ e^{-Mr} ,$$

with the constant

$$C = 2^{-\frac{5}{6}}\ e^{-\frac{1}{4}}\ A_G^3 = 0.921862\ldots .$$

The function $F^{(3)}$ in (6.5) gives the three-particle contribution to the correlation function. Using the definitions (1.3), (4.1) and the above relation, one can obtain:

$$\tilde{F}(p^2) = C\left\{1 - \frac{1}{\sqrt{1 + p^2/M^2}}\right\} + \ldots . \tag{6.6}$$

Here the dots stand for contributions of the massive multiparticle intermediate states. The last relation implies that the spectral density (6.2) can be written as

$$\Delta\tilde{F}(s) = \frac{C}{\pi}\ \frac{\Theta(s - M^2)}{\sqrt{s/M^2 - 1}} + \Theta(s - 9M^2)\ \Delta\tilde{F}^{(3)}(s) , \tag{6.7}$$

where

$$\Theta(s) = \begin{cases} 1 & \text{for } s \geq 0 \\ 0 & \text{for } s < 0 \end{cases} ,$$

and $\Delta\tilde{F}^{(3)}$ is some function which contributes to the spectral density only above the threshold $s = 9M^2$.

7. Numerics

In Table 1 we present results of numerical evaluation of the function F (1.3) as a function of the scaling distance Mr ($r = |x|$). To estimate the short-distance behavior, we use the perturbative expansion (3.31). As was already mentioned, the parameter $\bar{\tau}_0$ allows one to have control over the accuracy of the truncated series, so we calculate (3.31) for two different values of $\bar{\tau}_0$: -0.25 and $+0.25$. To determine the long-distance behavior of the function F, we use the formula (6.5), where the three-particle contribution $F^{(3)}$ was obtained by means of Eq. (5.2) along with formulas for the three-particle form factors quoted in [11] (see Appendix C). It is interesting to see that the sum of the one- and three-particle contributions to F is very near to unity at $r = 0$ (to within 1%), which indicates that this three-particle computation of the correlation function is in fact accurate to about 1% for all distance scales (more accurate, of course, for larger r). Also, note that the crossover between the long- and short-distance asymptotics appears to be at the scaling distances $Mr \sim 0.001 - 0.01$, where both asymptotics coincide to within about 0.1%.

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Mr	$F^{(1)}$	$F^{(3)}$	$F^{(1)}+F^{(3)}$	$F^{(pert)}$ $(\bar{\tau}_0=-0.25)$	$F^{(pert)}$ $(\bar{\tau}_0=0.25)$
0	.921862	.068	.990	1.00000	1.00000
.00001	.921853	.0553	.9771	.980129	.980130
.00005	.921816	.0522	.9740	.976311	.976314
.0001	.921770	.0504	.9722	.974192	.974196
.0002	.921678	.0483	.9700	.971674	.971678
.001	.920941	.0415	.9624	.963508	.963520
.002	.920020	.0375	.9575	.958435	.958454
.01	.912689	.0252	.9379	.939386	.939460
.025	.899101	.0168	.9159	.919294	.919494
.05	.876902	.0106	.8875	.894050	.894547
.075	.855251	.00738	.86263	.871796	.872717
.1	.834135	.00541	.83955	.850520	.852013
.15	.793454	.00317	.79662	.808380	.811548
.2	.754757	.00200	.75676	.765139	.770842
.25	.717947	.00131	.71926	.719980	.729252
.3	.682932	.000889	.683822	.672640	.686654
.35	.649625	.000617	.650243	.623153	.643171
.4	.617942	.000436	.618379	.571774	.599063
.45	.587805	.000313	.588118	.518942	.554677
.5	.559137	.000227	.559365	.465257	.510405

Table 1. The scaling function F (1.2), (1.3). The first columns give the results of the long-distance expansion which includes contributions of the one-, three- and one+three-particle states. The data in the last two columns correspond to the perturbative expansion (3.31) for the two different values of the auxiliary parameter $\bar{\tau}_0$.

Appendix A.

In this appendix, we give the first few terms in the expansion of $J_n(a,c)$ (3.6) around $c=-2$. The coefficients in this expansion involve standard functions of a, which could then easily be used to obtain an expansion of $J_n\left(\frac{2\omega}{1+\rho}, \frac{-2}{1+\rho}\right)$ in powers of ρ, as is needed in (3.23). To simplify the result, we will use the parameter

$$b = c + 2\,.$$

We find the following expansions in b of the functions $A(q,b-2)$, $B(q,b-2)$ (3.8) involved in (3.9):

$$
A(q,b-2) = \frac{\Gamma(2-b-q)\,\Gamma(b)}{\Gamma(2-b)\Gamma(1+b-q)}\ \Big\{1+b^2\Big(\frac{\pi^2}{6}+\frac{\psi(1-q)+\gamma_E}{q}\Big)+O(b^3)\Big\}\,,
$$

$$
B(q,b-2) = \frac{\Gamma(q+b)\Gamma(b)}{\Gamma(q)\Gamma(2b)}\Big\{1+\frac{b}{2q}+\frac{b^2}{2q^2}\big(q^2\psi'(q)-1\big)+\frac{b^3}{4q}\big(q\psi''(q)+2\psi'(q)\big)+O(b^4)\Big\}.
$$

Hence,

$$
\begin{aligned}
J_n(a,b-2) =& \frac{\pi^2}{2}\ \frac{4a^2-n^2+2nb}{b^2(1-b)^2}\ \exp\Big\{-G_n(a)\,b+\\
&\Big(\frac{G_n''(a)}{12}-2\ \frac{a\,G_n'(a)+G_n(a)}{n^2-4a^2}+\frac{10}{3}\,\zeta(3)\Big)\,b^3+O(b^4)\Big\}\ ,
\end{aligned}
$$

where

$$
G_n(a)=\psi(a+n/2)+\psi(-a+n/2)+2\gamma_E\ ,
$$

and $G_n'(a)=\frac{d}{da}\,G_n(a)\,,\quad G_n''(a)=\frac{d^2}{da^2}\,G_n(a)\,.$

Appendix B.

In this appendix, we write down explicit expressions for the coefficients u_1, u_2, v_1 and v_2 taking part in the expansion (3.16), (3.26) of the the function Γ_g.

On the one hand, from the assumption (5.6), the coefficients of odd powers of ρ in the exponential factor of $Z_{n,\omega}$ (3.28) are completely fixed by the conjectured constant $\mathbf{Z}_n(\omega\beta)$ (5.4). This fixes u_1, u_2 uniquely, giving

$$
\begin{aligned}
u_1 =& \Big(\omega^2-\frac{n^2}{16}\Big)\,\Big(T_n(2\omega)-\frac{3}{2}\Big)+\frac{n(n-2)}{16}\,,\\
u_2 =& \frac{1}{12}\Big(\omega^2-\frac{n^2}{16}\Big)\Big(\omega^2+\frac{n^2}{16}-\frac{1}{2}\Big)\,T_n''(2\omega)+\frac{\omega(4\omega^2-1)}{12}\,T_n'(2\omega)+\frac{1}{4}\Big(\omega^2-\frac{1}{12}\Big)T_n(2\omega)\\
&-\frac{n(n+4)}{768}-\frac{11\omega^2}{48}+\frac{1}{24}\,\bar\tau_0+\frac{\tau_2}{2}\,\Big(\omega^2-\frac{n^2}{16}\Big)\,,
\end{aligned}
\tag{B.1}
$$

where

$$
\begin{aligned}
&T_n(a)=\psi(a+n/2)+\psi(-a+n/2)+2\gamma_E+2\bar\tau_0\,,\\
&T_n'(a)=\frac{d}{da}\,T_n(a)\,,\qquad T_n''(a)=\frac{d^2}{da^2}\,T_n(a)\,.
\end{aligned}
$$

On the other hand, the expansion in powers of ρ of (3.23) uniquely determines the coefficients u_1, v_1 and $v_2-\frac{3}{2}\,u_2$ in the first equation of (3.26). The coefficient u_1 thus obtained is in agreement with (B.1), verifying the assumption (5.6) to first order. The coefficients v_1 and (using the expressions $u_{1,2}$ from (B.1)) v_2 are

$$
\begin{aligned}
v_1 =\ & \frac{\omega}{2}\left(\omega^2-\frac{n^2}{16}\right)T_n'(2\omega)+\frac{1}{4}\left(\omega^2-\frac{n^2}{16}\right)T_n^2(2\omega) \\
& -\frac{3}{4}\left(\omega^2-\frac{n(5n-4)}{48}\right)T_n(2\omega)+\frac{7}{8}\left(\omega^2-\frac{n(17n-20)}{112}\right)+\frac{u_1}{2}\ , \\
v_2 =\ & \left(\omega^2-\frac{n^2}{16}\right)\left(\frac{1}{24}-\frac{\omega^2}{4}\right)T_n''(2\omega) \\
& -\left(\left(\omega^2-\frac{n^2}{16}\right)\frac{T_n(2\omega)}{2}+\frac{1}{4}\left(\omega^2+\frac{n(n-4)}{16}-\frac{1}{2}\right)\right)\omega\, T_n'(2\omega) \\
& -\frac{1}{12}\left(\omega^2-\frac{n^2}{16}\right)T_n^3(2\omega)-\frac{1}{8}\left(\omega^2+\frac{n(n-4)}{16}\right)T_n^2(2\omega)-\frac{1}{8}\left(\omega^2-\frac{n(n-2)}{8}-\frac{1}{2}\right)T_n(2\omega) \\
& -\frac{1}{8}\left(\omega^2-\frac{n^2}{16}\right)\left(2\,\tau_2-14\,\zeta(3)-3\right)-\frac{n(n-8)}{256}+\frac{u_1}{8}+\frac{v_1}{2}+\frac{3u_2}{2}-\frac{\bar{\tau}_0}{8}\ .
\end{aligned}
$$

Appendix C.

In this appendix, we give the formula for the three-particle contribution $F^{(3)}$ (6.5) to the fermion two-point function in the $SU(2)$-Thirring model that we used for our numerical calculations. We first specialize the expression written in [11] to the case of three-particle form factors of the field $\mathcal{O}^1_{\beta/4}$ for $\beta^2=1$:

$$
\begin{aligned}
& \langle\, vac\,|\, \mathcal{O}^1_{1/4}(0)\,|\,A_-(\theta_1)\ldots A_+(\theta_k)\ldots A_-(\theta_3)\,\rangle_{in} = -\frac{A_G^{\frac{9}{2}}\,\Gamma^3(\frac{1}{4})}{2^{\frac{15}{4}}\,e^{\frac{3}{8}}\,\pi^{\frac{9}{4}}}\ e^{\frac{i\pi}{8}}\ M^{\frac{1}{4}}\times \\
& \prod_{m=1}^{3} e^{\frac{\theta_m}{4}}\ \prod_{m<j} G(\theta_m-\theta_j)\ \Big\{ \int_{C_+}\frac{d\gamma}{2\pi}\, e^{-\frac{\gamma}{2}}\prod_{p=1}^{k} W(\theta_p-\gamma)\prod_{p=k+1}^{3} W(\gamma-\theta_p)+ \\
& \int_{C_-}\frac{d\gamma}{2\pi}\, e^{-\frac{\gamma}{2}}\prod_{p=1}^{k-1} W(\theta_p-\gamma)\prod_{p=k}^{3} W(\gamma-\theta_p)\Big\}\ .
\end{aligned}
\tag{C.1}
$$

Here the functions G and W are

$$
G(\theta)=i\ \frac{2^{-\frac{5}{12}}\,e^{\frac{1}{4}}\,\Gamma(\frac{1}{4})}{\sqrt{\pi}\,A_G^3}\ \sinh(\theta/2)\ \exp\Big(\int_0^\infty\frac{dt}{t}\ \frac{\sinh^2 t(1-i\theta/\pi)\,e^{-t}}{\sinh(2t)\ \cosh(t)}\Big) \tag{C.2}
$$

and

$$
W(\theta)=2\ \frac{\Gamma(\frac{3}{4}-\frac{i\theta}{2\pi})\Gamma(-\frac{1}{4}+\frac{i\theta}{2\pi})}{\Gamma^2(\frac{1}{4})}\ . \tag{C.3}
$$

The contour C_+ starts from $-\infty$ on the real axis of the complex γ-plane, goes above the poles located at $\gamma=\theta_p+i\pi/2$, $p=1,\ldots,k$ and below those located at $\gamma=\theta_p-i\pi/2$,

$p = k+1, \ldots, 3$, always staying in the strip $-\pi/2 - 0 < \Im m\,\gamma < \pi/2 + 0$, and finally extends to $+\infty$ on the real axis. Similarly, the contour C_- goes above the poles located at $\gamma = \theta_p + i\pi/2$, $p = 1, \ldots, k-1$ and below those at $\gamma = \theta_p - i\pi/2$, $p = k, \ldots, 3$. Notice that the integrals in (C.1) can be expressed in terms of the generalized hypergeometric function ${}_3F_2$ at unity.

Using the expressions (C.1) and performing one of the rapidity integrals in (5.2), one can obtain the following form for the function $F^{(3)}$ in (6.5):

$$F^{(3)} = \frac{2\, e^{-\frac{3}{4}} A_G^9}{3\pi\, \Gamma^6(\frac{1}{4})} \int_{-\infty}^{\infty} dxdy\ \frac{e^{-Mr\sqrt{3+2\cosh x+2\cosh y+2\cosh(x-y)}}}{(3+2\cosh x+2\cosh y+2\cosh(x-y))^{\frac{1}{4}}} \times$$
$$\left(2\,|R_1(x,y)|^2 + |R_2(x,y)|^2\right)\, |G(x)G(y)G(x-y)|^2\, e^{\frac{x+y}{2}} \left(\frac{e^{-x}+e^{-y}+1}{e^x+e^y+1}\right)^{\frac{1}{4}} .$$

The functions R_1 and R_2 here are

$$R_2(x,y) = e^{-\frac{x}{2}+\frac{i\pi}{4}}\ R_1(-x, y-x) - e^{-\frac{y}{2}-\frac{i\pi}{4}}\ R_1^*(-y, x-y)$$

and

$$\begin{aligned} R_1(x,y) = &-\frac{\cosh\frac{x}{2}\cosh\frac{y}{2}}{2\sinh x \sinh y}\ U\Big(-\frac{1}{2}, -\frac{1}{2}-\frac{ix}{2\pi}, -\frac{1}{2}-\frac{iy}{2\pi}; -\frac{ix}{2\pi}, -\frac{iy}{2\pi}\Big) \\ &+ e^{-\frac{x}{2}}\frac{\cosh\frac{y-x}{2}\cosh\frac{x}{2}}{2\sinh(y-x)\sinh x}\ U\Big(\frac{1}{2}, \frac{1}{2}-\frac{i(y-x)}{2\pi}, \frac{1}{2}+\frac{ix}{2\pi}; 1-\frac{i(y-x)}{2\pi}, 2+\frac{ix}{2\pi}\Big) \\ &+ e^{-\frac{y}{2}}\frac{\cosh\frac{x-y}{2}\cosh\frac{y}{2}}{2\sinh(x-y)\sinh y}\ U\Big(\frac{1}{2}, \frac{1}{2}-\frac{i(x-y)}{2\pi}, \frac{1}{2}+\frac{iy}{2\pi}; 1-\frac{i(x-y)}{2\pi}, 2+\frac{iy}{2\pi}\Big)\ , \end{aligned}$$

where $U(a,b,c;d,e)$ is related to the generalized hypergeometric function ${}_3F_2$ by

$$U(a,b,c;d,e) = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(d)\Gamma(e)}\ {}_3F_2(a,b,c;d,e;1)\ .$$

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